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LETTER TO THE EDITOR

Mixed-spin Ising model on the union jack lattice

Adam Lipowski† and Tsuyoshi Horiguchi

Department of Computer and Mathematical Sciences, GSIS, Tohoku University, Sendai 980-77, Japan

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Abstract. We show that the mixed-spin Ising model on the union jack lattice with $S = \frac{1}{2}$ and $S = 1$ spin operators placed on different sublattices is equivalent to the eight-vertex model. On a two-dimensional manifold in a parameter space, the model corresponds to the symmetric eight-vertex model and thus is solvable. We locate exactly a coexistence surface between two ordered phases of the mixed-spin model. This surface is bounded by a bicritical line with continuously changing exponents.

Although understood in many respects, two-dimensional Ising models are still intensively studied and a lot of open problems still remain. Certainly, the behaviour of frustrated models and models with $S > \frac{1}{2}$ has not yet been fully elucidated.

Recently, some mixed-spin Ising models have been proposed as possible models of certain ferrimagnetics [1, 2]. In such models both frustration and a higher value of spin play an important role and approximate methods, which are reliable in other cases, might even give qualitatively wrong results. This is the case for the model on the square lattice where Monte Carlo simulations and transfer-matrix calculations [3, 4] ruled out the existence of the compensation point and the tricritical point which was predicted by the mean-field approximations [1, 2].

In this letter we consider a mixed-spin Ising model on the union jack lattice. Surprisingly, the presence of the additional interactions simplifies the problem and enables us to transform the model into a solvable eight-vertex model and examine some of its properties.

The Hamiltonian of this model is written as

$$H = -J_1 \sum \mu_i \mu_j - J_2 \sum \mu_i S_j - D \sum S_i^2 \qquad (1)$$

where $\mu_i = \pm 1$ and $S_i = \pm 1, 0$. The operators μ_i and S_i are located on eight- and four-coordinated sites, respectively (see figure 1). In the following we put $J_2 = 1$.

The ground-state structure of this model can be easily found by comparing the energies of the corresponding configurations. The result is shown in figure 2.

First, let us decimate spins S_i. Since spins S_i interact only with spins μ_i and the remaining (after decimation) spins μ_i are two-state variables, the resulting model is equivalent to the eight-vertex model with the following weights:

$$\omega_1 = e^{2\beta J_1}\left[1 + 2e^{\beta D}\cosh(4\beta)\right] \qquad \omega_2 = e^{-2\beta J_1}\left(1 + 2e^{\beta D}\right) \qquad (2)$$

$$\omega_3 = \omega_4 = 1 + 2e^{\beta D} \qquad \omega_5 = \omega_6 = \omega_7 = \omega_8 = 1 + 2e^{\beta D}\cosh(2\beta) \qquad (3)$$

† Permanent address: Department of Physics, A Mickiewicz University, Poznań, Poland.

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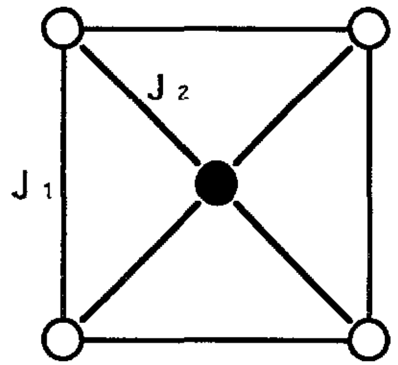

Figure 1. Elementary cell of the union jack lattice. Eight- and four-coordinated sites are denoted as open and full circles, respectively.

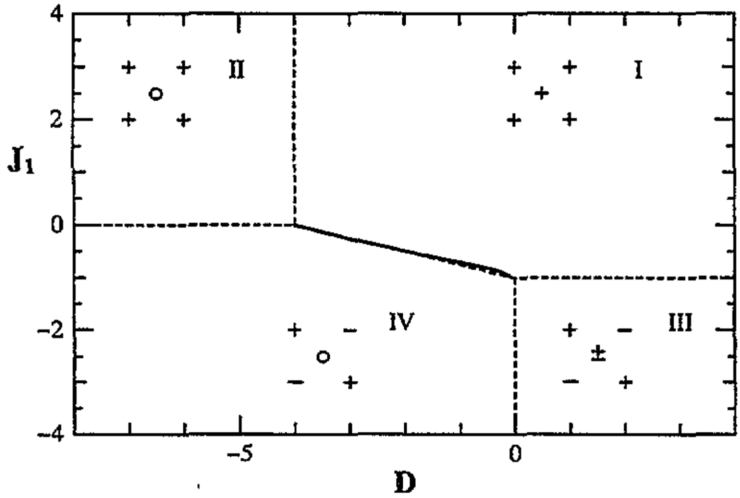

Figure 2. The ground-state structure of the model (1). In phase III the spins S_i can take values ± 1 arbitrarily. The full line between phases I and IV is a $T = 0$ projection of the critical line (6).

where $\beta = 1/T$ and we put $k_B = 1$. In equations (2) and (3) we follow the standard assignment between spin and vertex configurations [5].

One branch of solutions of the eight-vertex model corresponds to the so-called 'free-fermion' case [6]. In this case the weights ω_i have to satisfy the following equation:

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 . \tag{4}$$

Substituting the weights (2) and (3) into (4), we obtain that $e^{\beta D}[\cosh(2\beta) - 1]^2 = 0$ with the solutions $\beta = 0$ or $D = -\infty$. Both cases are rather uninteresting since the model (1) is then trivially equivalent to the $S = \frac{1}{2}$ Ising model. Let us notice that for the $S = \frac{1}{2}$ Ising model on the union jack lattice we can also perform such decimation and the resulting eight-vertex model satisfies the 'free-fermion' condition for arbitrary couplings. Thus one can easily rederive the results obtained by other more elaborate methods [7, 8].

The second branch of solutions corresponds to the symmetric case [5]. Since in our model we already have $\omega_3 = \omega_4$, $\omega_5 = \omega_6$ and $\omega_7 = \omega_8$, thus the symmetric case is obtained by imposing only one condition: $\omega_1 = \omega_2$ or equivalently

$$e^{2\beta J_1}\left[1 + 2e^{\beta D}\cosh(4\beta)\right] = e^{-2\beta J_1}\left(1 + 2e^{\beta D}\right) . \tag{5}$$

This condition describes a 2D manifold in the 3D parameter space (D, J_1, T). It is easy to realize that real solutions of (5) only exist for $-1 < J_1 < 0$. Thus, in the present approach, the model (1) is unsolvable on the square lattice (i.e. when $J_1 = 0$).

On the manifold (5) the vertex model becomes critical when $\omega_1 = \omega_3 + \omega_5 + \omega_7$ or equivalently

$$\left\{\left(1+2e^{\beta D}\right)\left[1+2e^{\beta D}\cosh(4\beta)\right]\right\}^{1/2} = 3 + 2e^{\beta D}[1+\cosh(2\beta)] \qquad (6)$$

where we eliminated J_1 using (5). The critical line obtained as a solution of (6) is shown in figure 2 (projection into the $T = 0$ plane) and in figure 3 (projection into the $J_1 = 0$ plane). Along this line critical exponents change continuously. For example, the exponent ν is given as

$$\nu^{-1} = \frac{4}{\pi}\arctan\frac{1+2e^{\beta D}\cosh(2\beta)}{[(1+2e^{\beta D})^3(1+2e^{\beta D}\cosh(4\beta))]^{1/2}} \qquad (7)$$

and is shown in figure 4 as a function of D.

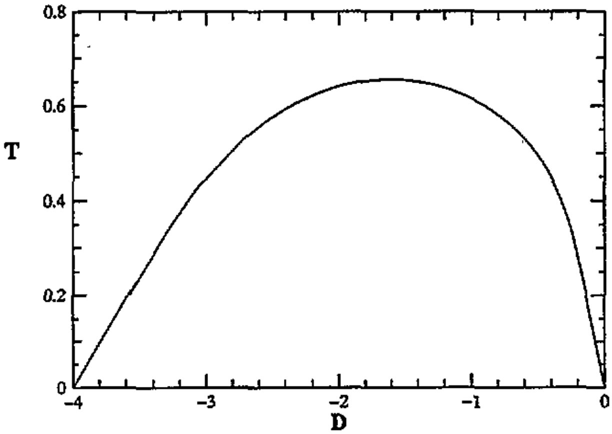

Figure 3. A $J_1 = 0$ projection of the critical line (6).

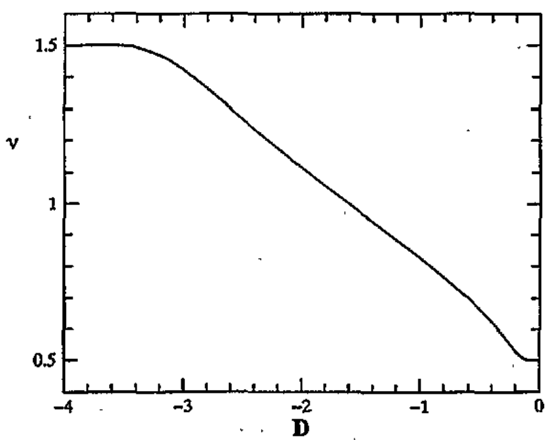

Figure 4. The exponent ν as a function of D.

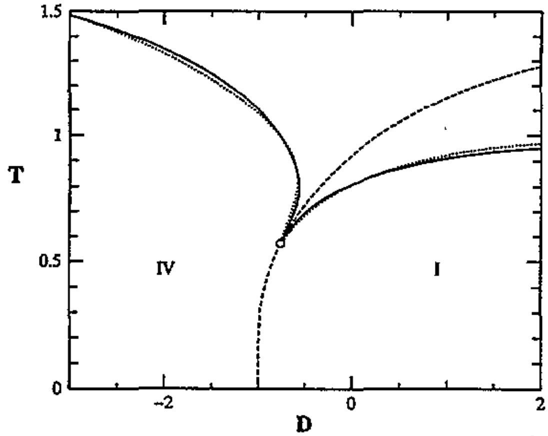

Figure 5. The phase diagram of the model (1) for $J_1 = -0.75$. The broken curve denotes the solvable manifold (5). The full and dotted curves denote continuous phase transitions obtained by the phenomenological renormalization group. Our calculations were performed for pairs of strips of widths (2, 4) (dotted) and (4, 6) (full). Below the bicritical point (circle) the broken curve denotes the first-order transition line between phases I and IV.

In the Ising model, as is well known, the condition that the magnetic field vanishes, actually describes the coexistence line between two phases of opposite magnetization. This first-order transition line ends at the critical point. The same situation appears in the model (1). The solvable manifold (5) corresponds to the symmetric eight-vertex model, i.e. to the model with zero electric field. On the other hand, as is easy to check, in the ordered phase of the vertex model this manifold crosses the $T = 0$ plane (of the parameter space) along the ground-state transition line $J_1 = -1 - D/4$. This enables us to identify phases on both sides of the manifold (5). Thus we arrive at the following conclusion: in the ordered phase of the vertex model (i.e. below the critical temperature given by (6)) this manifold gives the exact location of the first-order transitions between phases I and IV.

It seems plausible that off the coexistence line the ordered phases I and IV undergo a continuous phase transition to the paramagnetic phase and thus the non-universal critical point given by (6) is actually a bicritical point. This is confirmed by the numerical calculations using the phenomenological renormalization group [9]. Placing the vertex model with the weights (2) and (3) on a strip of finite width L we can find the critical point from the condition that the ratio of the correlation length and the width L is independent of the size. Although our calculations were not very extensive ($L \leqslant 6$), they clearly show (see figure 5) that the critical lines which separate phases I and IV from the paramagnetic phase indeed meet at the bicritical point. We also calculated the exponent ν along the critical lines and its value is close to unity. This indicates that along these lines the model (1) exhibits Ising-type critical behaviour. Let us notice that in the ordered phase the solvable manifold is nearly vertical and thus the critical points (6) are located almost above the line of the ground-state transitions.

Zero states of the $S = 1$ spin variable at the centre of the elementary cell can be interpreted as a non-magnetic impurity. It is already known [10, 11] that when such impurities are distributed in a certain periodic manner then the resulting model is equivalent to the sixteen-vertex model which satisfies the 'free-fermion' condition. Analysis of this model [10, 11] shows that its critical behaviour is of the Ising type and the ordered phases are always (except some limiting cases, e.g. $T = 0$) separated by the paramagnetic phase. Our results show that a random, annealed distribution of impurities introduces a qualitative difference: ordered phases might coexist even at finite temperatures and the critical behaviour might be non-universal.

The knowledge of the exact location of the coexistence surface and of the bicritical point can be useful in studying, e.g. crossover phenomena in the model (1) or in testing some approximate methods. Our analysis can be easily generalized to the case where at the four-coordinated sites we have arbitrary spin-S variables. We can also consider anisotropic couplings.

It might also be interesting to notice that represented in terms of two-state variables, the eight-vertex model has to contain rather peculiar four-body interactions or next-next-nearest neighbour interactions. We can see that introducing the three-state variables offers a new representation where only non-crossing interactions appear.

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